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# Anharmonic Corrections in Constant-Cutoff Soliton Model

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We suggest a quantum stabilization method for the $SU(2)$ $\sigma$-model, based on the constant-cutoff limit of the cutoff quantization method developed by Balakrishna *et al.*, which avoids the difficulties with the usual soliton boundary conditions pointed out by Iwasaki and Ohyama. We investigate the baryon number $B = 1$ sector of the model and show that after the collective coordinate quantization it admits a stable soliton solution which depends on a single dimensional arbitrary constant. We then derive the results for anharmonic corrections to the hyperon energy in the bound-state approach to the $SU(3)$-soliton model for the hyperons, with $SU(3)$-symmetry breaking. Thus we show that the anharmonic corrections give, as in the case of the complete Skyrme model, negative contributions to the hyperon energies and that they are of the same order of magnitude as those obtained using the complete Skyrme model for bound heavy-flavor two-meson systems in the case of cascade hyperons.

## 1. INTRODUCTION

It was shown by Skyrme (1961, 1962) that baryons can be treated as solitons of a nonlinear chiral theory. The original Lagrangian of the chiral $SU(2)$ $\sigma$-model is

$$\mathcal{L} = \frac{F_\pi^2}{16} \operatorname{Tr} \partial_\mu U \, \partial^\mu U^+ \tag{1.1}$$

where

$$U = \frac{2}{F_\pi} (\sigma + i\boldsymbol{\tau} \cdot \boldsymbol{\pi}) \tag{1.2}$$

is a unitary operator ($UU^+ = 1$) and $F_\pi$ is the pion-decay constant. In (1.2) $\sigma = \sigma(\mathbf{r})$ is a scalar meson field and $\boldsymbol{\pi} = \boldsymbol{\pi}(\mathbf{r})$ is the pion isotriplet.

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The classical stability of the soliton solution to the chiral $\sigma$-model Lagrangian requires an additional *ad hoc* term, proposed by Skyrme (1961, 1962), to be added to (1.1):

$$\mathcal{L}_{\mathrm{Sk}} = \frac{1}{32e^2}\,\mathrm{Tr}[U^+\,\partial_\mu U,\; U^+\,\partial_\nu U]^2 \tag{1.3}$$

with a dimensionless parameter $e$ and where $[A, B] = AB - BA$. It was shown by several authors [see Adkins *et al.* (1983), Witten (1979, 1983a,b), and, for extensive lists of additional references, Holzwarth and Schwesinger (1986) and Nyman and Riska (1990)] that, after collective quantization using the spherically symmetric ansatz

$$U_0(\mathbf{r}) = \exp[i\boldsymbol{\tau}\cdot\mathbf{r}_0 F(r)], \qquad \mathbf{r}_0 = \mathbf{r}/r \tag{1.4}$$

the chiral model with both (1.1) and (1.3) included gives good agreement with experiment for several important physical quantities. Thus it should be possible to derive the effective chiral Lagrangian as a sum of (1.1) and (1.3) from a more fundamental theory like QCD. On the other hand, it is not easy to generate a term like (1.3) and give a clear physical meaning to the dimensionless constant $e$ in (1.3) using QCD.

Mignaco and Wulck (1989) (MW) indicated therefore the possibility to build a stable single-baryon ($n = 1$) quantum state in the simple chiral theory with the Skyrme stabilizing term (1.3) omitted. They showed that the chiral angle $F(r)$ is in fact a function of a dimensionless variable $s = \frac{1}{2}\chi''(0)r$, where $\chi''(0)$ is an arbitrary dimensional parameter intimately connected to the usual stability argument against the soliton solution for the nonlinear $\sigma$-model Lagrangian.

Using the adiabatically rotated ansatz $U(\mathbf{r}, t) = A(t)U_0(\mathbf{r})A^+(t)$, where $U_0(\mathbf{r})$ is given by (1.4), MV obtained the total energy of the nonlinear $\sigma$-model soliton in the form

$$E = \frac{\pi}{4}F_\pi^2\,\frac{1}{\chi''(0)}\,a + \frac{1}{2}\,\frac{[\chi''(0)]^3}{(\pi/4)F_\pi^2 b}\,J(J+1) \tag{1.5}$$

where

$$a = \int_0^\infty \left[\frac{1}{4}\,s^2\left(\frac{d\mathcal{F}}{dS}\right)^2 + 8\,\sin^2\!\left(\frac{1}{4}\,\mathcal{F}\right)\right] ds \tag{1.6}$$

$$b = \int_0^\infty ds\,\frac{64}{3}\,s^2\,\sin^2\!\left(\frac{1}{4}\,\mathcal{F}\right) \tag{1.7}$$

and $\mathcal{F}(s)$ is defined by

$$F(r) = (s) = -n\pi + \tfrac{1}{4}\mathcal{F}(s) \tag{1.8}$$

The stable minimum of the function (1.5) with respect to the arbitrary dimensional scale parameter $\chi''(0)$ is

$$E = \frac{4}{3} F_\pi \left[ \frac{3}{2} \left( \frac{\pi}{4} \right)^2 \frac{a^3}{b} J(J + 1) \right]^{1/4} \tag{1.9}$$

Despite the nonexistence of the stable classical soliton solution to the nonlinear $\sigma$-model, it is possible, after the collective coordinate quantization, to build a stable chiral soliton at the quantum level, provided that there is a solution $F = F(r)$ which satisfies the soliton boundary conditions, i.e., $F(0) = -n\pi$, $F(\infty) = 0$, such that the integrals (1.6) and (1.7) exist.

However, as pointed out by Iwasaki and Ohyama (1989), the quantum stabilization method in the form proposed by MW is not correct since in the simple $\sigma$-model the conditions $F(0) = -n\pi$ and $F(\infty) = 0$ cannot be satisfied simultaneously. In other words, if the condition $F(0) = -\pi$ is satisfied, Iwasaki and Ohyama obtained numerically $F(\infty) \to -\pi/2$, and the chiral phase $F = F(r)$ with correct boundary conditions does not exist.

Iwasaki and Ohyama also proved analytically that both boundary conditions $F(0) = -n\pi$ and $F(\infty) = 0$ cannot be satisfied simultaneously. Introducing a new variable $y = 1/r$ into the differential equation for the chiral angle $F = F(r)$, we obtain

$$\frac{d^2F}{dy^2} = \frac{1}{y^2} \sin 2F \tag{1.10}$$

There are two kinds of asymptotic solutions to equation (1.10) around the point $y = 0$, which is called a regular singular point if $\sin 2F \approx 2F$. These solutions are

$$F(y) = \frac{m\pi}{2} + cy^2, \qquad m = \text{even integer} \tag{1.11}$$

$$F(y) = \frac{m\pi}{2} + \sqrt{cy} \cos\left[ \frac{\sqrt{7}}{2} \ln(cy) + \alpha \right], \qquad m = \text{odd integer} \tag{1.12}$$

where $c$ is an arbitrary constant and $\alpha$ is a constant to be chosen appropriately. When $F(0) = -n\pi$, we want to know which of these two solutions is approached by $F(y)$ when $y \to 0$ ($r \to \infty$). In order to answer that question we multiply (1.10) by $y^2F'(y)$, integrate with respect to $y$ from $y$ to $\infty$, and use $F(0) = -n\pi$. Thus we get

$$y^2F'(y) + \int_y^\infty 2y[F'(y)]^2 \, dy = 1 - \cos[2F(y)] \tag{1.13}$$

Since the left-hand side of (1.13) is always positive, the value of $F(y)$ is always limited to the interval $n\pi - \pi < F(y) < n\pi + \pi$. Taking the limit $y \to 0$, we find that (1.13) is reduced to

$$\int_0^\infty 2y[F'(y)]^2\, dy = 1 - (-1)^m \tag{1.14}$$

where we used (1.11)–(1.12). Since the left-hand side of (1.14) is strictly positive, we must choose an odd integer $m$. Thus the solution satisfying $F(0) = -n\pi$ approaches (1.12) and we have $F(\infty) \neq 0$. The behavior of the solution (1.11) in the asymptotic region $y \to \infty$ $(r \to 0)$ is investigated by multiplying (1.10) by $F'(y)$, integrating from 0 to $y$, and using (1.11). The result is

$$[F'(y)]^2 = \frac{2\sin^2 F(y)}{y^2} + \int_0^y \frac{2\sin^2 F(y)}{y^3}\, dy \tag{1.15}$$

From (1.15) we see that $F'(y) \to \text{const}$ as $y \to \infty$, which means that $F(r) \simeq 1/r$ for $r \to 0$. This solution has a singularity at the origin and cannot satisfy the usual boundary condition $F(0) = -n\pi$.

In Dalarsson (1991a,b, 1992), I suggested a method to resolve this difficulty by introducing a radial modification phase $\varphi = \varphi(r)$ in the ansatz (1.4) as follows

$$U(\mathbf{r}) = \exp[i\boldsymbol{\tau}\cdot\mathbf{r}_0 F(r) + i\varphi(r)], \qquad \mathbf{r}_0 = \mathbf{r}/r \tag{1.16}$$

Such a method provides a stable chiral quantum soliton, but the resulting model is an entirely noncovariant chiral model, different from the original chiral $\sigma$-model.

In the present paper we use the constant-cutoff limit of the cutoff quantization method developed by Balakrishna *et al.* (1991; see also Jain *et al.*, 1989) to construct a stable chiral quantum soliton within the original chiral $\sigma$-model. Then we apply this method to derive the results for anharmonic corrections to the hyperon energy in the bound-state approach to the $SU(3)$-soliton model for the hyperons, with $SU(3)$-symmetry breaking. Thus we show that the anharmonic corrections give, as in the case of the complete Skyrme model, negative contributions to the hyperon energies and that they are of the same order of magnitude as those obtained using the complete Skyrme model for bound heavy-flavor two-meson systems in the case of cascade hyperons.

The reason why the cutoff approach to the problem of chiral quantum solitons works is connected to the fact that the solution $F = F(r)$ which satisfies the boundary condition $F(\infty) = 0$ is singular at $r = 0$. From the physical point of view the chiral quantum model is not applicable to the region about the origin, since in that region there is a quark-dominated bag of the soliton.

However, as argued in Balakrishna *et al.* (1991), when a cutoff $\epsilon$ is introduced, then the boundary conditions $F(\epsilon) = -n\pi$ and $F(\infty) = 0$ can be satisfied. Balakrishna *et al.* (1991) discussed an interesting analogy with the damped pendulum, showing clearly that as long as $\epsilon > 0$, there is a chiral phase $F = F(r)$ satisfying the above boundary conditions. The asymptotic forms of such a solution are given by equation (2.2) in Balakrishna *et al.* (1991). From these asymptotic solutions we immediately see that for $\epsilon \to 0$ the chiral phase diverges at the lower limit.

Different applications of the constant-cutoff approach have been discussed in Dalarsson (1993, 1995a–c).

## 2. CONSTANT-CUTOFF STABILIZATION

Substituting (1.4) into (1.1), we obtain for the static energy of the chiral baryon

$$E_0 = \frac{\pi}{2} F_\pi^2 \int_{\epsilon(t)}^{\infty} dr \left[ r^2 \left(\frac{dF}{dr}\right)^2 + 2 \sin^2 F \right] \tag{2.1}$$

In (2.1) we avoid the singularity of the profile function $F = F(r)$ at the origin by introducing the cutoff $\epsilon(t)$ at the lower boundary of the space interval $r \in [0, \infty]$, i.e., by working with the interval $r \in [\epsilon, \infty]$. The cutoff itself is introduced, following Balakrishna *et al.* (1991), as a dynamic time-dependent variable.

From (2.1) we obtain the following differential equation for the profile function $F = F(r)$:

$$\frac{d}{dr}\left(r^2 \frac{dF}{dr}\right) = \sin 2F \tag{2.2}$$

with the boundary conditions $F(\epsilon) = -\pi$ and $F(\infty) = 0$, such that the correct soliton number is obtained. The profile function $F = F[r; \epsilon(t)]$ now depends implicitly on time $t$ through $\epsilon(t)$. Thus in the nonlinear $\sigma$-model Lagrangian

$$L = \frac{F_\pi^2}{16} \int \mathrm{Tr}(\partial_\mu U\, \partial^\mu U^+)\, d^3\mathbf{r} \tag{2.3}$$

we use the ansätze

$$U(\mathbf{r}, t) = A(t) U_0(\mathbf{r}, t) A^+(t) \tag{2.4}$$
$$U^+(\mathbf{r}, t) = A(t) U_0^+(\mathbf{r}, t) A^+(t)$$

where

$$U_0(\mathbf{r}, t) = \exp\{i\boldsymbol{\tau}\cdot\mathbf{r}_0 F[r, \epsilon(t)]\} \tag{2.5}$$

The static part of the Lagrangian (2.3), i.e.,

$$L = \frac{F_\pi^2}{16}\int \mathrm{Tr}(\nabla U\cdot\nabla U^+)\, d^3\mathbf{r} = -E_0 \tag{2.6}$$

is equal to minus the energy $E_0$ given by (2.1). The kinetic part of the Lagrangian is obtained using (2.4) with (2.5) and it is equal to

$$L = \frac{F_\pi^2}{16}\int \mathrm{Tr}(\partial_0 U\, \partial_0 U^+)\, d^3\mathbf{r} = bx^2\, \mathrm{Tr}[\partial_0 A\, \partial_0 A^+] + c[\dot{x}(t)]^2 \tag{2.7}$$

where

$$b = \frac{2\pi}{3} F_\pi^2 \int_1^\infty \sin^2 F\, y^2\, dy, \qquad c = \frac{2\pi}{9} F_\pi^2 \int_1^\infty y^2\left(\frac{dF}{dy}\right)^2 y^2\, dy \tag{2.8}$$

with $x(t) = [\epsilon(t)]^{3/2}$ and $y = r/\epsilon$. On the other hand, the static energy functional (2.1) can be rewritten as

$$E_0 = ax^{2/3}, \qquad a = \frac{\pi}{2} F_\pi^2 \int_1^\infty \left[y^2\left(\frac{dF}{dy}\right)^2 + 2\sin^2 F\right] dy \tag{2.9}$$

Thus the total Lagrangian of the rotating soliton is given by

$$L = c\dot{x}^2 - ax^{2/3} + 2bx^2\dot{\alpha}_\nu\dot{\alpha}^\nu \tag{2.10}$$

where $\mathrm{Tr}(\partial_0 A\, \partial_0 A^+) = 2\dot{\alpha}_\nu\dot{\alpha}^\nu$ and $\alpha_\nu$ ($\nu = 0, 1, 2, 3$) are the collective coordinates defined as in (Bhaduri, 1988). In the limit of a time-independent cutoff ($\dot{x} \to 0$) we can write

$$H = \frac{\partial L}{\partial\dot{\alpha}^\nu}\dot{\alpha}^\nu - L = ax^{2/3} + 2bx^2\dot{\alpha}_\nu\dot{\alpha}^\nu = ax^{2/3} + \frac{1}{2bx^2} J(J+1) \tag{2.11}$$

where $\langle \mathbf{J}^2\rangle = J(J+1)$ is the eigenvalue of the square of the soliton angular momentum. A minimum of (2.11) with respect to the parameter $x$ is reached at

$$x = \left[\frac{2}{3}\frac{ab}{J(J+1)}\right]^{-3/8} \Rightarrow \epsilon^{-1} = \left[\frac{2}{3}\frac{ab}{J(J+1)}\right]^{1/4} \tag{2.12}$$

The energy obtained by substituting (2.12) into (2.11) is given by

$$E = \frac{4}{3}\left[\frac{3}{2}\frac{a^3}{b}J(J+1)\right]^{1/4} \tag{2.13}$$

This result is identical to the result obtained by Mignaco and Wulck, which is easily seen if we rescale the integrals $a$ and $b$ in such a way that $a \to \frac{1}{4}\pi F_\pi^2 a$ and $b \to \frac{1}{4}\pi F_\pi^2 b$ and introduce $f_\pi = 2^{-3/2}F_\pi$. However in the present approach, as shown in Balakrishna *et al.* (1991), there is a profile function $F = F(y)$ with proper soliton boundary conditions $F(1) = -\pi$ and $F(\infty) = 0$ and the integrals $a$, $b$, and $c$ in (2.9)–(2.10) exist and are shown in Balakrishna *et al.* (1991) to be $a = 0.78$ GeV$^2$, $b = 0.91$ GeV$^2$, and $c = 1.46$ GeV$^2$ for $F_\pi = 186$ MeV.

Using (2.13), we obtain the same prediction for the mass ratio of the lowest states as Mignaco and Wulck (1989), which agrees rather well with the empirical mass ratio for the $\Delta$ resonance and the nucleon. Furthermore, using the calculated values for the integrals $a$ and $b$, we obtain the nucleon mass $M(N) = 1167$ MeV, which is about 25% higher than the empirical value of 939 MeV. However, if we choose the pion-decay constant equal to $F_\pi = 150$ MeV, we obtain $a = 0.507$ GeV$^2$ and $b = 0.592$ GeV$^2$, giving exact agreement with the empirical nucleon mass.

Finally it is of interest to know how large the constant cutoffs are for the above values of the pion-decay constant in order to check if they are in the physically acceptable ballpark. Using (2.12), it is easily shown that for the nucleons ($J = 1/2$) the cutoffs are equal to

$$\epsilon = \begin{cases} 0.22\ \text{fm} & \text{for} \quad F_\pi = 186\ \text{MeV} \\ 0.27\ \text{fm} & \text{for} \quad F_\pi = 150\ \text{MeV} \end{cases} \tag{2.14}$$

From (2.14) we see that the cutoffs are too small to agree with the size of the nucleon (0.72 fm), as we should expect, since the cutoffs rather indicate a size of the quark-dominated bag in the center of the nucleon. Thus we find that the cutoffs are of reasonable physical size. Since the cutoff is proportional to $F_\pi^{-1}$, we see that the pion-decay constant must be less than 57 MeV in order to obtain a cutoff which exceeds the size of the nucleon. Such values of pion-decay constant are not relevant to any physical phenomena.

## 3. THE *SU*(3)-EXTENDED CONSTANT-CUTOFF MODEL

### 3.1. The Effective Interaction

The Lagrangian density of the bound-state model of hyperons is, with the Skyrme stabilizing term omitted, given by (Dalarsson, 1993, 1991a–c; Callan and Klebanov, 1985; Callan *et al.*, 1988; Pari *et al.*, 1991)

$$\mathcal{L} = \frac{F_\pi^2}{16} \mathrm{Tr}\, \partial_\mu U\, \partial^\mu U^+ + \frac{F_\pi^2 m_\pi^2 + 2F^2 m^2}{48} \mathrm{Tr}(U + U^+ - 2)$$
$$+ \frac{F_\pi^2 m_\pi^2 - F^2 m^2}{24} \mathrm{Tr}[\sqrt{3}\, \lambda_8 (U + U^+)]$$
$$- \frac{F_\pi^2 - F^2}{48} \mathrm{Tr}[(1 - \sqrt{3}\, \lambda_8)(U\, \partial_\mu U^+\, \partial^\mu U + U^+\, \partial_\mu U\, \partial^\mu U^+)] \qquad (3.1)$$

where $m_\pi$ and $m$ are pion and heavy-flavor meson ($K$, $D$, or $B$) masses, respectively, and $F$ is the heavy-flavor meson ($K$, $D$, or $B$) decay constant with the empirical ratios to the pion-decay constant $F_K/F_\pi \approx 1.23$, $F_D/F_\pi \approx 2.4$, and $F_B/F_\pi \approx 2.8$.

The first term in (3.1) is the usual $\sigma$-model Lagrangian, given by (1.1), while the remaining three terms are all chiral- and flavor-symmetry-breaking terms present in the mesonic sector of the model. All flavor-symmetry-breaking terms in the effective Lagrangian (3.1) also break the chiral symmetry just as quark-mass terms do in the underlying QCD Lagrangian. In addition to the action obtained using the Lagrangian (3.1), the Wess–Zumino action in the form

$$S_{\mathrm{WZ}} = -\frac{iN_c}{240\pi^2} \int d^5x\, e^{\mu\nu\alpha\beta\gamma}$$
$$\times \mathrm{Tr}[(U^+\, \partial_\mu U)(U^+\, \partial_\nu U)(U^+\, \partial_\alpha U)(U^+\, \partial_\beta U)(U^+\, \partial_\gamma U)] \qquad (3.2)$$

must be included into the total action of the system, where $N_c$ is the number of colors in the underlying QCD. The Wess–Zumino action defines the topological properties of the model important for the quantization of the solitons. In the $SU(2)$ case the Wess–Zumino action vanishes identically and therefore was not present in the discussions of Sections 1 and 2.

In the present approach the meson-soliton field is written in the form

$$U = \sqrt{U_\pi}\, U_K\, \sqrt{U_\pi} \qquad (3.3)$$

where $U_\pi$ is a $SU(3)$ extension of the usual $SU(2)$ skyrmion field used to describe the nucleon spectrum and $U_K$ is the field describing the heavy-flavor mesons

$$U_\pi = \begin{bmatrix} u_\pi & 0 \\ 0 & 1 \end{bmatrix}, \qquad U_K = \exp\left\{ i\, \frac{2^{3/2}}{F_\pi} \begin{bmatrix} 0 & K \\ K^+ & 0 \end{bmatrix} \right\} \qquad (3.4)$$

In (3.4) $u_\pi$ is the usual $SU(2)$-skyrmion field given by (1.4). The two-dimensional vector $K$ in (3.5) is the heavy-flavor meson doublet

$$K = \begin{bmatrix} K^+ \\ K^0 \end{bmatrix}, \quad \begin{bmatrix} \bar{D}^0 \\ D^- \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} B^+ \\ B^0 \end{bmatrix} \tag{3.5}$$

$$K^+ = [K^- \quad \bar{K}^0], \quad [D^0 \quad D^+], \quad \text{or} \quad [B^- \quad \bar{B}^0] \tag{3.6}$$

We now substitute (3.3), with $U_\pi$ and $U_K$ defined by (3.4), into the total action of the kaon–soliton system and expand $U_K$ to fourth order in the heavy-flavor-meson fields (3.5) and (3.6), to obtain

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} \tag{3.7}$$

where

$$\mathcal{L}^{(0)} = \frac{F_\pi^2}{16} \operatorname{Tr} \partial_\mu u_\pi \, \partial^\mu u_\pi^+ \tag{3.8}$$

is the usual Lagrangian density of the $SU(2)$ soliton. Because of the form of the Lagrangian (3.1) only the terms of even order in the heavy-flavor-meson field $K$ appear in (3.7) and the second-order term is given by

$$\mathcal{L}^{(2)} = \partial_\mu K \, \partial^\mu K^+ + K^+ \left[ \frac{1}{4} \left( \frac{dF}{dr} \right)^2 + \frac{\cos F \,(1 - \cos F)}{r^2} - \frac{1 - \cos F}{2r^2} \boldsymbol{\tau} \cdot \mathbf{L} \right] K$$
$$- m^2 K^+ K + \frac{N_c}{4f^2} B_\mu [K^+ D^\mu K - (D^\mu K)^+ K] \tag{3.9}$$

where $B_\mu$ is the anomalous baryon current of the soliton

$$B_\mu = \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} \operatorname{Tr}[(U^+ \partial^\nu U)(U^+ \partial^\alpha U)(U^+ \partial^\beta U)] \tag{3.10}$$

the time component of which is given by

$$B_0 = \frac{\sin^2 F}{2\pi^2 r^2} \frac{dF}{dr} \tag{3.11}$$

and $D^\mu$ is defined by

$$D^\mu = \partial^\mu + \tfrac{1}{2}[(U_\pi^+)^{1/2} \, \partial^\mu (U_\pi)^{1/2} + (U_\pi)^{1/2} \, \partial^\mu (U_\pi^+)^{1/2}] \tag{3.12}$$

The mode decomposition of the heavy-flavor-meson field gives the following wave equation for the bound meson:

$$\nabla^2 K(\mathbf{r}) + \left[ v_0(r) - 2\,\frac{1 - \cos F}{r^2}\,\mathbf{I}\cdot\mathbf{L} \right] K(\mathbf{r}) - m^2 K(\mathbf{r})$$

$$+ 2\omega\lambda(r)K(\mathbf{r}) + \omega^2 K(\mathbf{r}) = 0 \tag{3.13}$$

where

$$v_0 = \frac{1}{4}\left(\frac{dF}{dr}\right)^2 + \frac{\cos F\,(1 - \cos F)}{r^2} \tag{3.14}$$

$$\lambda(r) = -\frac{N_c}{2\pi^2 F^2}\,\frac{\sin^2 F}{r^2}\,\frac{dF}{dr} \tag{3.15}$$

As argued in Dalarsson (1993, 1995a–c) and Björnberg *et al.* (1995), the ground-state solution to (3.13) is a *P*-state and it is described by a wave function of the type

$$K(\mathbf{r}) = A\,\frac{k(r)}{\sqrt{4\pi}}\,\boldsymbol{\tau}\cdot\mathbf{r}_0\chi \tag{3.16}$$

where $k(r)$ is the radial wave function and $A$ is the rotation operator which transforms the meson isospin operator into an effective spin operator (Pari *et al.*, 1991).

Using (3.16), we obtain the radial wave equation for the lowest bound-state wave function $u_0 = rk_p(r)$ in the form

$$\frac{d^2 u_0}{dr^2} - v_{\text{eff}}(r)u_0 + [\omega^2 - m_K^2 + 2\omega\lambda(r)]u_0 = 0 \tag{3.17}$$

where $\omega$ is the lowest bound-state energy, the bound-state modes are normalized according to

$$8\pi \int_\epsilon^\infty dr\, r^2[\omega + \lambda(r)]k_p^*(r)k_p(r) = 1 \tag{3.18}$$

due to the form of the harmonic Lagrangian of the meson, and

$$v_{\text{eff}}(r) = -\frac{1}{4}\left[\left(\frac{dF}{dr}\right)^2 + 2\,\frac{\sin^2 F}{r^2}\right] + \frac{2}{r^2}\cos^4\frac{F}{2} \tag{3.19}$$

The fourth-order Lagrangian is given by

$$\mathcal{L}^{(4)} = \frac{2}{3F^2}\left\{4\left[m^2 + \frac{1 - \cos F}{r^2} - \frac{1}{4}\left(\frac{dF}{dr}\right)^2\right](K^+K)^2 - 4K^+K\,\partial_\mu K^+\,\partial^\mu K\right.$$

$$- 4(\partial_\mu K^+ K)(K^+\partial^\mu K) + (K^+\partial_\mu K)(K^+\partial^\mu K) + (\partial_\mu K^+ K)(\partial^\mu K^+ K)$$

$$+ 2i \frac{1 - \cos F}{r} K^+K[\nabla K^+ \cdot (\boldsymbol{\tau} \times \mathbf{r}_0)K - K^+(\boldsymbol{\tau} \times \mathbf{r}_0) \cdot \nabla K]$$

$$- 3i \frac{1 - \cos F}{r} [K^+\nabla K - \nabla K^+K] \cdot K^+(\boldsymbol{\tau} \times \mathbf{r}_0)K$$

$$- \frac{3}{2} K^+\left[\frac{\sin F}{r}\boldsymbol{\tau} + \frac{1 - \cos F}{r}(\boldsymbol{\tau} \times \mathbf{r}_0) + \left(\frac{dF}{dr} - \frac{\sin F}{r}\right)(\boldsymbol{\tau}\cdot\mathbf{r}_0)\mathbf{r}_0\right]K$$

$$\times K^+\left[\frac{\sin F}{r}\boldsymbol{\tau} + \frac{1 - \cos F}{r}(\boldsymbol{\tau} \times \mathbf{r}_0) + \left(\frac{dF}{dr} - \frac{\sin F}{r}\right)(\boldsymbol{\tau}\cdot\mathbf{r}_0)\mathbf{r}_0\right]K\Bigg\}$$

$$+ \frac{F_\pi^2}{2F^4}\left\{\left[\frac{1 - \cos F}{r^2} - \frac{1}{4}\left(\frac{dF}{dr}\right)^2\right](K^+K)^2 + K^+K(\partial_\mu K^+)(\partial^\mu K)\right.$$

$$+ (\partial_\mu K^+K)(K^+\partial^\mu K)$$

$$+ \frac{i}{2}\frac{1 - \cos F}{r} K^+K[\nabla K^+ \cdot (\boldsymbol{\tau} \times \mathbf{r}_0)K - K^+(\boldsymbol{\tau} \times \mathbf{r}_0) \cdot \nabla K]$$

$$+ \frac{i}{2}\frac{1 - \cos F}{r} [K^+\nabla K - \nabla K^+K] \cdot K^+(\boldsymbol{\tau} \times \mathbf{r}_0)K$$

$$- \frac{1}{2} K^+\left[\frac{\sin F}{r}\boldsymbol{\tau} + \frac{1 - \cos F}{r}(\boldsymbol{\tau} \times \mathbf{r}_0) + \left(\frac{dF}{dr} - \frac{\sin F}{r}\right)(\boldsymbol{\tau}\cdot\mathbf{r}_0)\mathbf{r}_0\right]K$$

$$\times K^+\left[\frac{\sin F}{r}\boldsymbol{\tau} + \frac{1 - \cos F}{r}(\boldsymbol{\tau} \times \mathbf{r}_0) + \left(\frac{dF}{dr} - \frac{\sin F}{r}\right)(\boldsymbol{\tau}\cdot\mathbf{r}_0)\mathbf{r}_0\right]K\Bigg\} \tag{3.20}$$

For a two-meson system with both mesons of the same flavor in the $P$-state (3.16), the mesons form a triplet state such that $\boldsymbol{\tau}\cdot\mathbf{L} = -2$ and $\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2 = 1$, and the Hamiltonian obtained from the anharmonic Lagrangian (3.20) gives the fairly simple end result for the anharmonic energy correction

$$\Delta E^{(4)} = -\frac{4}{3\pi F^2}\int_\epsilon^\infty r^2\,dr\,k^4(r)\left(\left\{m^2 + \frac{5}{2}\omega^2 + \frac{3}{2}\left[\frac{d}{dr}\ln k(r)\right]^2 - \frac{3}{8}\left(\frac{dF}{dr}\right)^2\right.\right.$$

$$\left. + \frac{3}{2r^2}\cos F\,(1 + \cos F)\right\} + \frac{F_\pi^2}{F^2}\left\{-\frac{3}{2}\omega^2 + \frac{3}{2}\left[\frac{d}{dr}\ln k(r)\right]^2\right.$$

$$\left.\left. + \frac{1}{4}\left(\frac{dF}{dr}\right)^2 + \frac{2}{r^2}\cos F\,(1 + \cos F)\right\}\right) \tag{3.21}$$

Introducing the dimensionless variable $y = r/\epsilon$ as well as the dimensionless parameters $\mu = \epsilon m$ and $\varphi = \epsilon\omega$, we obtain

$$\Delta E^{(4)} = -\frac{4\epsilon}{3\pi F^2}\int_1^\infty y^2\,dy\,k^4(y)\Bigg(\Bigg\{\mu^2 + \frac{5}{2}\varphi^2 + \frac{3}{2}\left[\frac{d}{dy}\ln k(y)\right]^2 - \frac{3}{8}\left(\frac{dF}{dy}\right)^2$$
$$+ \frac{3}{2y^2}\cos F\,(1 + \cos F)\Bigg\} + \frac{F_\pi^2}{F^2}\Bigg\{-\frac{3}{2}\varphi^2 + \frac{3}{2}\left]\frac{d}{dy}\ln k(r)\right]^2$$
$$+ \frac{1}{4}\left(\frac{dF}{dy}\right)^2 + \frac{2}{y^2}\cos F\,(1 + \cos F)\Bigg\}\Bigg) \tag{3.22}$$

Using the solutions for $F(y)$ and $k(y)$ corresponding to the Lagrangian (3.1), we obtain the numerical values for the anharmonic corrections for cascade hyperons as two-meson states with both mesons in the ground $P$-state shown in Table I. The values obtained using the constant-cutoff approach are compared to those obtained using the complete Skyrme model in Björnberg *et al.* (1995). Table I shows that there is a general qualitative agreement. However, the present approach offers much simpler algebra by avoiding the lengthy and painful calculations of the contributions from the Skyrme stabilizing term (1.3) or its equivalent given by equation (2.4) in Björnberg *et al.* (1995). It also eliminates the discussion about the alternative choices of the Skyrme stabilizing term (Pari *et al.*, 1991; Björnberg *et al.*, 1995). Björnberg *et al.* (1995) showed that a choice other than (1.3) was necessary to make it possible to carry out the difficult calculations, but such a choice gives nonnegligible errors in the heavy-flavor-meson bound-state energies and it is concluded that the calculations with the original Skyrme stabilizing term (1.3) are more realistic. In our case the constant-cutoff approach gives fairly good results for the spectra and magnetic moments of strange hyperons, using the second-order Lagrangian (3.9) with the Skyrme stabilizing term omitted (Dalarsson, 1993, 1995a–c). The quartic corrections, as in the case of the complete Skyrme model (Björnberg *et al.*, 1995), are shown to be

**Table I.** Numerical Values of Quartic Contributions to the Energy of a Two-Meson System with Both Mesons in the Ground State for Different Meson Families ($K$, $D$, $B$)

| | $\Delta E^{(4)}$ (MeV) | |
|---|---|---|
| Meson family | This work | Björnberg *et al.* (1995) |
| $K$ | −37 | −26 |
| $D$ | −271 | −233 |
| $B$ | −733 | −639 |

small, amounting to a few percent of the predicted masses of the cascade hyperons. They can therefore be considered as insignificant. As in Björnberg *et al.* (1995) we have neglected the quartic contribution of the Wess–Zumino action (3.2), which splits the energies of hyperons with opposite flavors and contributes a smaller amount to the effective interaction than the terms in (3.20). As argued in Björnberg *et al.* (1995), the relative importance of the Wess–Zumino term is smaller in the case of the heavy-flavor hyperons.

From the results for the bound-state energies obtained in the harmonic approximation for two-meson states, we see that the quartic corrections given in Table I amount to $<9\%$ of the total bound-state energies, corresponding to $<2\%$ of the predicted energies of the cascade particles.

## 4. CONCLUSIONS

We have shown the possibility of using the Skyrme model for the calculation of the anharmonic corrections to the hyperon energy in the bound-state approach to the $SU(3)$-soliton model for hyperons without the use of the Skyrme stabilizing term proportional to $e^{-2}$, which makes the practical calculations very complicated and introduces the problem of the choice of the stabilizing term such that the harmonic and anharmonic contributions are calculated in a compatible and empirically correct way.

For such a simple model with only one arbitrary dimensional constant $F_\pi$, which is chosen equal to its empirical value $F_\pi = 186$ MeV, we show that the anharmonic corrections give, as in the case of the complete Skyrme model, negative contributions to the hyperon energies and that they are of the same order of magnitude as those obtained using the complete Skyrme model for bound heavy-flavor two-meson systems in the case of cascade hyperons. The numerical results are in general qualitative agreement with and of the same accuracy as those obtained using the complete Skyrme model (Björnberg *et al.*, 1995).

Furthermore, the uncertainties in the choice of the form of the fourth-order stabilizing term as well as the convenient values of the parameters $F_\pi$ and $e$ in it are eliminated altogether.

## REFERENCES

Adkins, G. S., Nappi, C. R., and Witten, E. (1983). *Nuclear Physics B*, **228**, 552.
Balakrishna, B. S., Sanyuk, V., Schechter, J., and Subbaraman, A. (1991). *Physical Review D*, **45**, 344.
Bhaduri, R. K. (1988). *Models of the Nucleon*, Addison-Wesley, Reading, Massachusetts.
Björnberg, M., Dannbom, K., and Riska, D. O. (1995). *Nuclear Physics A*, **582**, 621.
Callan, C. G., and Klebanov, I. (1985). *Nuclear Physics B*, **262**, 365.

Callan, C. G., Hornbostel, K., and Klebanov, I. (1988). *Physics Letters B*, **202**, 269.
Dalarsson, N. (1991a). *Modern Physics Letters A*, **6**, 2345.
Dalarsson, N. (1991b). *Nuclear Physics A*, **532**, 708.
Dalarsson, N. (1992). *Nuclear Physics A*, **536**, 573.
Dalarsson, N. (1993). *Nuclear Physics A*, **554**, 580.
Dalarsson, N. (1995a). *International Journal of Theoretical Physics*, **34**(1), 81.
Dalarsson, N. (1995b). *International Journal of Theoretical Physics*, **34**(6), 949.
Dalarsson, N. (1995c). *International Journal of Theoretical Physics*, **34**(10), 2129.
Holzwarth, G., and Schwesinger, B. (1986). *Reports on Progress in Physics*, **49**, 825.
Iwasaki, M., and Ohyama, H. (1989). *Physical Review*, **40**, 3125.
Jain, P., Schechter, J., and Sorkin, R. (1989). *Physical Review D*, **39**, 998.
Mignaco, J. A., and Wulck, S. (1989). *Physical Review Letters*, **62**, 1449.
Nyman, E. M., and Riska, D. O. (1990). *Reports on Progress in Physics*, **53**, 1137.
Pari, G., Schwesinger, B., Walliser, H. (1991). *Physics Letters B*, **255**, 1.
Skyrme, T. H. R. (1961). *Proceedings of the Royal Society A*, **260**, 127.
Skyrme, T. H. R. (1962). *Nuclear Physics*, **31**, 556.
Witten, E. (1979). *Nuclear Physics B*, **160**, 57.
Witten, E. (1983a). *Nuclear Physics B*, **223**, 422.
Witten, E. (1983b). *Nuclear Physics B*, **223**, 433.